

Mathematics Extension 2

Worked solutions and marking guide

Year 12 | 2017 syllabus | Independent Mock A

Keep this document separate until the timed attempt is complete.

Equivalent correct methods earn full credit. Marking criteria describe mathematical evidence; avoid double penalties for a single carried error where later reasoning remains valid.

Answer key

1	2	3	4	5	6	7	8	9	10
B	C	A	D	B	A	D	C	B	D

Mark and item totals

Section I: 10 marks. Section II: six 15-mark questions, 37 marked items, 90 marks. Total: 100 marks. The longer 4- or 5-mark items are Q13(d), Q14(f) and Q16(e).

Binding scope: 2026 HSC, Mathematics Extension 2 Stage 6 Syllabus (2017), including prescribed Advanced and Extension 1 assumed knowledge. The 2024 syllabus was consulted for continuity only; its first HSC examination is in 2027.

Question 1 - B (1 mark)

The negation of a universal assertion is an existential assertion. For a fixed x , the negation of $P \Rightarrow Q$ is $P \wedge \neg Q$. Here P is $x^2 < 9$ and Q is $f(x) > 0$, so a counterexample requires $x^2 < 9$ and $f(x) \leq 0$. Hence B.

Syllabus: MEX12-2, MEX12-8; MEX-P1 The Nature of Proof: formal language, negation, implication and quantifiers.

Question 2 - C (1 mark)

Write $z = x + iy$. Then $z + 2\bar{z} = 3x - iy$, so $x = 3$ and $y = -2$. Thus

$$(1 + i)z = (1 + i)(3 - 2i) = 5 + i.$$

Its imaginary part is 1. Hence C.

Syllabus: MEX12-1, MEX12-4; MEX-N1, N1.1 Arithmetic of complex numbers: Cartesian form, equality, conjugates and multiplication.

Question 3 - A (1 mark)

Expanding the scalar product gives

$$|2\mathbf{a} + \mathbf{b}|^2 = 4|\mathbf{a}|^2 + 4\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 = 16 - 12 + 9 = 13.$$

Hence A. The data are consistent, since $\cos \theta = -3/(2 \cdot 3) = -1/2$.

Syllabus: MEX12-3; MEX-V1, V1.2 Further operations with three-dimensional vectors: magnitude and scalar product.

Question 4 - D (1 mark)

The integrand is $2/(x + 1) + 1/(x + 2)$. Therefore the integral is

$$[2 \ln(x + 1) + \ln(x + 2)]_0^1 = 2 \ln 2 + \ln 3 - \ln 2 = \ln 6.$$

Hence D.

Syllabus: MEX12-5; MEX-C1 Further Integration: partial fractions and definite rational integrals.

Question 5 - B (1 mark)

In SHM, acceleration has the form $a = -n^2(x - c)$. Subtract the two observations:

$$-10 - 6 = -n^2(6 - 2).$$

Hence $n^2 = 4$ and $n = 2$, so the period is $2\pi/n = \pi$ seconds. Hence B. The equilibrium position is $c = 7/2$ metres, though it is not needed.

Syllabus: MEX12-6; MEX-M1, M1.1 Simple harmonic motion: acceleration-displacement equation and period.

Question 6 - A (1 mark)

Since $(a - b)^2 \geq 0$ and $a + b = 4$, we have $ab \leq 4$. The expression equals

$$1 + \frac{2(a + b) + 4}{ab} = 1 + \frac{12}{ab} \geq 4.$$

Equality holds at $a = b = 2$, so 4 is an attained minimum. Hence A.

Syllabus: MEX12-2, MEX12-8; MEX-P1 The Nature of Proof: non-negative squares and the arithmetic-mean/geometric-mean inequality.

Question 7 - D (1 mark)

Put $z = x + iy$. Subtracting $(x - 1)^2 + y^2 = 4$ from $(x + 1)^2 + y^2 = 12$ gives $4x = 8$, so $x = 2$ and $y^2 = 3$. Thus $z_1 = 2 + i\sqrt{3}$ and $z_2 = 2 - i\sqrt{3}$, in either order. Their sum is 4 and their product is 7. Consequently

$$z_1^3 + z_2^3 = (z_1 + z_2)^3 - 3z_1z_2(z_1 + z_2) = 64 - 84 = -20.$$

Hence D.

Syllabus: MEX12-4, MEX12-7; MEX-N2, N2.2 Geometrical implications of complex numbers: subsets defined by modulus relations; MEX-N1 arithmetic and powers.

Question 8 - C (1 mark)

The first two coordinates give $x^2 + y^2 = 4$. Therefore

$$2 = (x + y)^2 = x^2 + y^2 + 2xy = 4 + 2xy,$$

so $xy = -1$. Also $xy = 4 \sin t \cos t = 2 \sin 2t$. Thus

$$z = 3 + \sin 2t = 3 + \frac{xy}{2} = \frac{5}{2}.$$

Hence C. The constraint is attainable, for example at $t = -\pi/12$; both admissible coordinate pairs give the same height.

Syllabus: MEX12-3, MEX12-7; MEX-V1, V1.3: vector equations of parametric curves in two or three dimensions; Extension 1 corequisite double-angle identities.

Question 9 - B (1 mark)

Integrate by parts with $u = \ln(1 + x)$ and $dv = (1 + x)^{-2}dx$. Then $du = dx/(1 + x)$ and $v = -(1 + x)^{-1}$. The integral equals

$$\left[-\frac{\ln(1 + x)}{1 + x} \right]_0^1 + \int_0^1 (1 + x)^{-2}dx = -\frac{\ln 2}{2} + \frac{1}{2} = \frac{1 - \ln 2}{2}.$$

Hence B.

Syllabus: MEX12-5; MEX-C1 Further Integration: integration by parts and definite integrals.

Question 10 - D (1 mark)

Put $c = k/m > 0$ and take $x = 0$ at the start of the first interval. From the supplied speed, $dt/dx = e^{cx}/u$. Therefore the elapsed time to reach displacement x is $t(x) = (e^{cx} - 1)/(cu)$. Put $q = e^{cd} > 1$. Successive equal-distance interval times are

$$\frac{q - 1}{cu}, \quad \frac{q(q - 1)}{cu}, \quad \frac{q^2(q - 1)}{cu}.$$

Their common ratio is q . The first two times give $q = 3$, so the third interval takes $3(3T) = 9T$. Hence D.

Syllabus: MEX12-6, MEX12-7; MEX-M1, M1.3 Resisted motion: horizontal motion under resistance proportional to a power of speed; velocity versus displacement and displacement versus time.

Question 11 (15 marks)

Short problems

(a) 2 marks

In polar form, $2 - 2i = 2\sqrt{2} \operatorname{cis}(-\pi/4)$. Thus

$$(2 - 2i)^8 = (2\sqrt{2})^8 \operatorname{cis}(-2\pi) = 4096.$$

Marking: Correct polar form or an equivalent exact repeated-squaring calculation. | Correct exact value 4096.

2017 mapping: MEX12-1, MEX12-4, MEX12-7, MEX12-8; MEX-N1.2, MEX-N2.1.

(b) 2 marks

Write $z = x + iy$. Squaring gives $(x - 1)^2 + y^2 = 4[x^2 + (y + 2)^2]$, or $3x^2 + 3y^2 + 2x + 16y + 15 = 0$. Hence

$$\left(x + \frac{1}{3}\right)^2 + \left(y + \frac{8}{3}\right)^2 = \frac{20}{9}.$$

The centre is $(-1/3, -8/3)$ and the radius is $2\sqrt{5}/3$.

Marking: Correct Cartesian equation. | Correct centre and radius.

2017 mapping: MEX12-1, MEX12-4, MEX12-7, MEX12-8; MEX-N2.2.

(c) 2 marks

For a perpendicular vector (x, y, z) , the scalar products give $2x - y + 2z = 0$ and $x + 2y = 0$. One solution is $(-4, 2, 5)$, whose magnitude is $\sqrt{45} = 3\sqrt{5}$. A required unit vector is

$$\frac{1}{3\sqrt{5}}(-4, 2, 5).$$

Its negative is equally valid. No cross product is needed.

Marking: Use two zero scalar products to obtain a nonzero perpendicular vector. | Correct normalisation.

2017 mapping: MEX12-3, MEX12-7, MEX12-8; MEX-V1.2.

(d) 2 marks

Use $x = \frac{1}{2}(2x + 2) - 1$ and $x^2 + 2x + 5 = (x + 1)^2 + 4$. The result is

$$\frac{1}{2}\ln(x^2 + 2x + 5) - \frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C.$$

The quadratic is positive for every real x .

Marking: Separate a logarithmic derivative and the completed-square integral. | Correct two terms and an arbitrary constant.

2017 mapping: MEX12-1, MEX12-5, MEX12-7, MEX12-8; MEX-C1.

(e) 2 marks

The equation is $\ddot{x} = -9(x - 2)$. Thus the equilibrium is $x = 2$ metres, the angular frequency is 3 radians per second, and the period is $2\pi/3$ seconds.

Marking: Equilibrium $x=2$. | Period $2\pi/3$ seconds.

2017 mapping: MEX12-6, MEX12-7, MEX12-8; MEX-M1.1.

(f) 2 marks

Among three consecutive integers, one is divisible by 3. At least one is even, so the product is divisible by 2 as well. Since 2 and 3 are relatively prime, the product is divisible by 6.

Marking: Explain both divisibility by 2 and by 3. | Justify divisibility by 6 for all positive integers.

2017 mapping: MEX12-1, MEX12-2, MEX12-7, MEX12-8; MEX-P1.

(g) 3 marks

For $n = 0$, the expression is 0, agreeing with u_0 . Suppose the formula holds for $n = k \geq 0$. Then

$$u_{k+1} = \frac{1 + 2(3^k - 1)/(3^k + 1)}{2 + (3^k - 1)/(3^k + 1)} = \frac{3^{k+1} - 1}{3^{k+1} + 1}.$$

The denominators are nonzero: under the hypothesis $0 \leq u_k < 1$, so $2 + u_k > 0$. Induction proves the result.

Marking: Verify the base case and state the induction hypothesis. | Substitute into the recurrence and simplify correctly. | Establish the required next case and conclude; denominators are valid.

2017 mapping: MEX12-1, MEX12-2, MEX12-7, MEX12-8; MEX-P2.

Question 12 (15 marks)

A chord and a constrained family of spheres

(a) 2 marks

Here $\overrightarrow{AB} = (5, 2, -3)$, of length $\sqrt{38}$, and the direction vector has length 3. Their scalar product is 18. Thus

$$\theta = \cos^{-1} \left(\frac{6}{\sqrt{38}} \right).$$

Marking: Correct scalar product and magnitudes. | Correct exact acute angle.

2017 mapping: MEX12-3, MEX12-7, MEX12-8; MEX-V1.2.

(b) 2 marks

At parameter t , the vector from the line to B is $(5 - 2t, 2 - t, -3 + 2t)$. Orthogonality gives

$$2(5 - 2t) + (2 - t) - 2(-3 + 2t) = 18 - 9t = 0.$$

Hence $t = 2$ and $H = (5, 0, -3)$. This is the minimum because the squared distance is a strictly convex quadratic, as shown next.

Marking: Correct orthogonality equation. | $t=2$ and $H=(5,0,-3)$.

2017 mapping: MEX12-3, MEX12-7, MEX12-8; MEX-V1.2, MEX-V1.3.

(c) 2 marks

Write $\overrightarrow{BX} = \overrightarrow{BH} + \overrightarrow{HX}$. Now $\overrightarrow{BH} = (-1, 0, -1)$, $\overrightarrow{HX} = (t - 2)(2, 1, -2)$, and their scalar product is zero. Therefore

$$|\overrightarrow{BX}|^2 = 2 + 9(t - 2)^2.$$

Direct expansion is also valid.

Marking: Correct perpendicular decomposition or expansion. | Required identity.

2017 mapping: MEX12-3, MEX12-7, MEX12-8; MEX-V1.2, MEX-V1.3.

(d) 2 marks

At an intersection, $9(t - 2)^2 + 2 = 11$, so $t = 1$ or 3 . Consequently

$$P = (3, -1, -1), \quad Q = (7, 1, -5).$$

Marking: Correct parameter equation and roots. | Correct ordered coordinates.

2017 mapping: MEX12-3, MEX12-7, MEX12-8; MEX-V1.3.

(e) 2 marks

The midpoint of PQ is H , and BH is perpendicular to PQ . Also $|PQ| = 2|(2, 1, -2)| = 6$ and $|BH| = \sqrt{2}$. Hence the area is $\frac{1}{2}(6)(\sqrt{2}) = 3\sqrt{2}$ square units.

Marking: Correct base and perpendicular height. | Area $3\sqrt{2}$.

2017 mapping: MEX12-3, MEX12-7, MEX12-8; MEX-V1.2, MEX-V1.3.

(f) 2 marks

Since $(1, -2, 0) \cdot (2, 1, -2) = 0$ and $(1, 0, 1) \cdot (2, 1, -2) = 0$, the vector

$$\overrightarrow{HC_\lambda} = (1 + \lambda, -2\lambda, 1)$$

is perpendicular to the chord. Since $HP = HQ = 3$, Pythagoras gives $|C_\lambda P|^2 = |HC_\lambda|^2 + 9 = |C_\lambda Q|^2$.

Marking: Establish orthogonality to the chord. | Use the midpoint or equal squared distances to prove the claim for all λ .

2017 mapping: MEX12-3, MEX12-7, MEX12-8; MEX-V1.2, MEX-V1.3.

(g) 3 marks

By (f), every real λ gives such a sphere. Its squared radius is

$$R^2 = 9 + (1 + \lambda)^2 + 4\lambda^2 + 1 = 5\left(\lambda + \frac{1}{5}\right)^2 + \frac{54}{5}.$$

The unique minimum occurs at $\lambda = -1/5$. Thus

$$R_{\min} = \sqrt{\frac{54}{5}}, \quad C_{-1/5} = \left(\frac{29}{5}, \frac{2}{5}, -2\right).$$

This minimises within the stated family of centres, not among all possible spheres through P and Q.

Marking: Obtain $R^2 = 5\lambda^2 + 2\lambda + 11$. | Justify the global minimum at $\lambda = -1/5$. | Correct radius and centre.

2017 mapping: MEX12-3, MEX12-7, MEX12-8; MEX-V1.2, MEX-V1.3.

Question 13 (15 marks)

Integration and a cumulative integral

(a)(i) 2 marks

Write the fraction as $A/(x+1) + (Bx+C)/(x^2+4)$. Coefficient comparison gives $A+B=1$, $B+C=3$, $4A+C=7$, hence $A=1$, $B=0$, $C=3$. Thus

$$\frac{x^2+3x+7}{(x+1)(x^2+4)} = \frac{1}{x+1} + \frac{3}{x^2+4}.$$

Marking: Correct partial-fraction form or coefficient equations. | Correct decomposition.

2017 mapping: MEX12-1, MEX12-5, MEX12-7, MEX12-8; MEX-C1.

(a)(ii) 2 marks

Integrate the decomposition:

$$\left[\ln(x+1) + \frac{3}{2} \tan^{-1}(x/2) \right]_0^2 = \ln 3 + \frac{3\pi}{8}.$$

No denominator vanishes on this interval.

Marking: Correct antiderivative. | Correct endpoint evaluation.

2017 mapping: MEX12-1, MEX12-5, MEX12-7, MEX12-8; MEX-C1.

(b) 3 marks

The limits become 0 and $\pi/4$. On this interval, $\sec \theta > 0$, so the integral becomes

$$\int_0^{\pi/4} \frac{\tan^2 \theta}{\sec \theta} d\theta = \int_0^{\pi/4} (\sec \theta - \cos \theta) d\theta.$$

Evaluating $\ln(\sec \theta + \tan \theta) - \sin \theta$ at the endpoints gives

$$\ln(1 + \sqrt{2}) - \frac{1}{\sqrt{2}}.$$

Marking: Correct substitution, transformed limits and sign of secant. | Reduce to sec theta-cos theta and integrate. | Correct exact value.

2017 mapping: MEX12-1, MEX12-5, MEX12-7, MEX12-8; MEX-C1.

(c) 2 marks

The substitution gives

$$I = \int_0^{\pi/2} (\pi/2 - u) \sin u \cos u du.$$

Add this to the original expression: $2I = (\pi/2) \int_0^{\pi/2} \sin x \cos x dx = \pi/4$. Thus $I = \pi/8$.

Marking: Correct transformed integral and symmetry equation. | Exact value $\pi/8$.

2017 mapping: MEX12-1, MEX12-5, MEX12-7, MEX12-8; MEX-C1.

(d) 4 marks

Let $K(a) = \int_0^a e^{-x} \cos 2x \, dx$. Integration by parts gives

$$J(a) = -e^{-a} \sin 2a + 2K(a).$$

A second integration by parts gives

$$K(a) = 1 - e^{-a} \cos 2a - 2J(a).$$

Substitute and rearrange: $5J(a) = 2 - e^{-a} \sin 2a - 2e^{-a} \cos 2a$, establishing the result. Both lower-endpoint terms are included.

Marking: First correct integration-by-parts relation. | Second correct relation. | Correct endpoint terms and equation in J. | Solve for the stated expression.

2017 mapping: MEX12-1, MEX12-5, MEX12-7, MEX12-8; MEX-C1.

(e) 2 marks

By the fundamental theorem, $J'(a) = e^{-a} \sin 2a$. It is positive on $(0, \pi/2)$ and negative on $(\pi/2, \pi)$. Thus the unique maximum is at $a = \pi/2$, and

$$J_{\max} = \frac{2}{5}(1 + e^{-\pi/2}).$$

Marking: Correct derivative and sign change, including the given domain. | Correct location and maximum value.

2017 mapping: MEX12-1, MEX12-5, MEX12-7, MEX12-8; MEX-C1, Advanced differentiation.

Question 14 (15 marks)

A vertical flight with quadratic resistance

(a) 2 marks

Take upwards as positive. During ascent both gravity and resistance act downwards, so

$m dv/dt = -mg - mgv^2/c^2$. Since $dy/dt = v$, the chain rule gives $dv/dt = v dv/dy$. Dividing by m gives the required equation.

Marking: Correct signs in Newton's law. | Correct chain-rule conversion to acceleration as a function of height.

2017 mapping: MEX12-6, MEX12-7, MEX12-8; MEX-M1.3.

(b) 2 marks

Separate variables: $v dv/(v^2 + c^2) = -(g/c^2) dy$. Integrating from $(y, v) = (0, u)$ gives

$$\frac{1}{2} \ln \frac{v^2 + c^2}{u^2 + c^2} = -\frac{gy}{c^2}.$$

Exponentiating and rearranging proves the result.

Marking: Correct separated integral and initial condition. | Correct exponential relation.

2017 mapping: MEX12-6, MEX12-7, MEX12-8; MEX-M1.3, MEX-C1.

(c) 2 marks

At the apex $v = 0$, so (b) gives

$$h = \frac{c^2}{2g} \ln \left(1 + \frac{u^2}{c^2} \right).$$

Using $dv/dt = -g(1 + v^2/c^2)$ during the ascent,

$$t_u = \frac{c^2}{g} \int_0^u \frac{dv}{c^2 + v^2} = \frac{c}{g} \tan^{-1}(u/c).$$

Marking: Correct maximum height. | Correct ascending time.

2017 mapping: MEX12-6, MEX12-7, MEX12-8; MEX-M1.3, MEX-C1.

(d) 2 marks

Take downwards as positive during descent. Then $w dw/dx = g(1 - w^2/c^2)$, with $w = 0$ at $x = 0$.

Integration yields

$$-\frac{1}{2} \ln \frac{c^2 - w^2}{c^2} = \frac{gx}{c^2},$$

giving the result. In particular $0 \leq w < c$ at every finite x , so the logarithm is well defined.

Marking: Correct descending equation and initial condition. | Correct speed-distance relation.

2017 mapping: MEX12-6, MEX12-7, MEX12-8; MEX-M1.3, MEX-C1.

(e) 3 marks

Set $x = h$ in (d) and use (c). Then

$$w_0 = \frac{uc}{\sqrt{u^2 + c^2}}.$$

Now $dw/dt = g(1 - w^2/c^2)$, so partial fractions give

$$t_d = \frac{c}{2g} \ln \frac{c + w_0}{c - w_0}.$$

With $s = \sqrt{u^2 + c^2}$, the ratio equals $(s + u)/(s - u) = (s + u)^2/c^2$. Hence

$$t_d = \frac{c}{g} \ln \left(\frac{u + \sqrt{u^2 + c^2}}{c} \right).$$

Marking: Correct returning speed with the positive square root. | Correct separated time integral and logarithmic expression in w_0 . | Simplify to the required parameters.

2017 mapping: MEX12-6, MEX12-7, MEX12-8; MEX-M1.3, MEX-C1.

(f) 4 marks

From (e), $c/\sqrt{u^2 + c^2} = 1/2$, so $c = u/\sqrt{3}$. Thus

$$T = t_u + t_d = \frac{u}{\sqrt{3}g} \left\{ \frac{\pi}{3} + \ln(2 + \sqrt{3}) \right\}.$$

For the comparison, define for $q \geq 0$

$$D(q) = \ln(q + \sqrt{1 + q^2}) - \tan^{-1} q.$$

Then $D(0) = 0$, and for $q > 0$,

$$D'(q) = \frac{1}{\sqrt{1 + q^2}} - \frac{1}{1 + q^2} > 0.$$

Therefore $D(\sqrt{3}) > 0$, proving $\ln(2 + \sqrt{3}) > \pi/3$, and hence $t_d > t_u$. Equivalently compare the corresponding integrands directly. The same argument proves the time inequality for every positive launch speed.

Marking: $c=u/\sqrt{3}$. | Correct total time. | Valid exact comparison through a derivative or integral inequality. | Strict inequality with a valid domain and conclusion; decimal evidence alone is insufficient.

2017 mapping: MEX12-6, MEX12-7, MEX12-8; MEX-M1.3, MEX-P1, MEX-C1.

Question 15 (15 marks)

A rectangle of constant area

(a) 2 marks

From $(p + iq)^2 = a + i$, $p^2 - q^2 = a$ and $2pq = 1$. Also $p^2 + q^2 = |a + i| = \sqrt{a^2 + 1}$. Hence

$$p^2 = \frac{\sqrt{a^2 + 1} + a}{2}, \quad q^2 = \frac{\sqrt{a^2 + 1} - a}{2}.$$

Both are strictly positive for all real a , establishing the required first-quadrant root.

Marking: Correct real/imaginary relations and modulus relation. | Correct two squared values.

2017 mapping: MEX12-1, MEX12-4, MEX12-7, MEX12-8; MEX-N1.1, MEX-N2.1.

(b) 2 marks

The equation is $(z^2 - a)^2 = -1$, so $z^2 = a + i$ or $a - i$. Its four distinct roots are

$$p + iq, \quad p - iq, \quad -p + iq, \quad -p - iq.$$

Their Cartesian coordinates are $(\pm p, \pm q)$. The side lengths are $2p$ and $2q$, giving area $4pq = 2$.

Marking: All four distinct roots. | Rectangle with area 2, justified from its sides.

2017 mapping: MEX12-1, MEX12-4, MEX12-7, MEX12-8; MEX-N2.1, MEX-N2.2.

(c) 2 marks

The diagonal length is

$$2\sqrt{p^2 + q^2} = 2(a^2 + 1)^{1/4}.$$

Since $(p + q)^2 = p^2 + q^2 + 2pq = \sqrt{a^2 + 1} + 1$, the perimeter is

$$L(a) = 4\sqrt{\sqrt{a^2 + 1} + 1}.$$

Marking: Correct diagonal length. | Correct perimeter.

2017 mapping: MEX12-1, MEX12-4, MEX12-7, MEX12-8; MEX-N2.2.

(d) 3 marks

The equation is $\sqrt{a^2 + 1} = s^2 - 1$. The left side is at least 1. Thus there are no solutions if $0 < s < \sqrt{2}$.

If $s = \sqrt{2}$, the only solution is $a = 0$. If $s > \sqrt{2}$, squaring is reversible because $s^2 - 1 > 0$, giving

$$a^2 = s^4 - 2s^2, \quad a = \pm s\sqrt{s^2 - 2}.$$

These two values are distinct and satisfy the original equation.

Marking: Establish the threshold $s \geq \sqrt{2}$ without accepting extraneous roots. | Correct real solution formula. | Complete classification, including the boundary.

2017 mapping: MEX12-1, MEX12-4, MEX12-7, MEX12-8; MEX-N2.1, MEX-P1.

(e) 3 marks

Suppose the perimeter is rational and let $s = L/4$, also rational. Then $r = s^2 - 1 = \sqrt{a^2 + 1}$ is rational and $r^2 = a^2 + 1$ is an integer. By the supplied fact, r is a positive integer. Since $r > |a|$, the two integers $r - a$ and $r + a$ are positive and satisfy

$$(r - a)(r + a) = 1.$$

Thus both are 1, giving $r = 1, a = 0$. It follows that $s^2 = 2$, impossible for rational s . Hence the perimeter is irrational for every integer a , including zero.

Marking: Correct contradiction assumption and show $\sqrt{a^2 + 1}$ is an integer. | Use integer factorisation to force $a=0$. | Complete the contradiction using the irrationality of $\sqrt{2}$.

2017 mapping: MEX12-1, MEX12-2, MEX12-7, MEX12-8; MEX-P1, MEX-N2.1.

(f) 3 marks

Write $\theta = \text{Arg}(z_a - i)$. Its real part p is positive, so $-\pi/2 < \theta < \pi/2$, an interval where tangent is strictly increasing. Since $q = 1/(2p)$,

$$\tan \theta = \frac{q - 1}{p} = \frac{1}{2p^2} - \frac{1}{p} = \frac{1}{2} \left(\frac{1}{p} - 1 \right)^2 - \frac{1}{2} \geq -\frac{1}{2}.$$

Equality holds exactly when $p = 1$, giving $q = 1/2$ and $a = p^2 - q^2 = 3/4$. Hence the minimum principal argument is $-\tan^{-1}(1/2)$, attained only at $a = 3/4$. All positive p occur in the family because $a = p^2 - 1/(4p^2)$, so the equality point is admissible.

Marking: Correct argument interval and tangent expression. | Prove a global lower bound and its equality condition. | Correct minimum argument and the unique parameter value.

2017 mapping: MEX12-1, MEX12-4, MEX12-7, MEX12-8; MEX-N1.2, MEX-N2.2, MEX-P1.

Question 16 (15 marks)

An alternating approximation and its first correction

(a) 3 marks

Put $q = x(1 - x)$ and

$$M_n = \int_0^1 q^n (2x - 1)^2 dx.$$

For $n \geq 0$,

$$\frac{d}{dx} \{(2x - 1)q^{n+1}\} = 2q^{n+1} - (n + 1)(2x - 1)^2 q^n.$$

Both boundary values vanish because $n + 1 \geq 1$. Integrating from 0 to 1 gives

$$(n + 1)M_n = 2B_{n+1}.$$

Also $(2x - 1)^2 = 1 - 4q$, so

$$M_n = B_n - 4B_{n+1}.$$

Combining these equations yields the auxiliary relation

$$B_{n+1} = \frac{n + 1}{2(2n + 3)} B_n,$$

and hence

$$M_n = \frac{B_n}{2n + 3}.$$

The argument includes $n = 0$. The auxiliary relation and $B_0 = 1$ also show that all B_n , and therefore all A_n , are rational; a factorial expression is not required.

Marking: Differentiate the suggested expression correctly. | Integrate with justified zero boundary terms to obtain $(n+1)M_n = 2B_{n+1}$. | Use $M_n = B_n - 4B_{n+1}$ to eliminate B_{n+1} and establish the required identity, including $n=0$.

2017 mapping: MEX12-1, MEX12-5, MEX12-7, MEX12-8; MEX-C1, MEX-P1, MEX-P2.

(b) 3 marks

The substitution gives

$$F_n = \int_0^1 \frac{[u(1 - u)]^n}{2 - u} du.$$

Rename u as x , add this to the original definition, and use $(1 + x)(2 - x) = 2 + x(1 - x)$:

$$2F_n = \int_0^1 [x(1 - x)]^n \left(\frac{1}{1 + x} + \frac{1}{2 - x} \right) dx = 3 \int_0^1 \frac{[x(1 - x)]^n}{2 + x(1 - x)} dx.$$

Writing $q = x(1 - x)$ and using $q/(2 + q) = 1 - 2/(2 + q)$,

$$F_{n+1} = \frac{3}{2} \int_0^1 \frac{q^{n+1}}{2 + q} dx = \frac{3}{2} B_n - 3 \int_0^1 \frac{q^n}{2 + q} dx = \frac{3}{2} B_n - 2F_n.$$

Marking: Correct substitution, integrand and limits. | Combine the two integral forms to obtain the symmetric denominator. | Establish the recurrence by algebra under the integral.

2017 mapping: MEX12-1, MEX12-2, MEX12-5, MEX12-7, MEX12-8; MEX-C1, MEX-P1, MEX-P2.

(c) 2 marks

For $n = 0$, $A_0 = 0$ and $F_0 = \int_0^1 (1+x)^{-1} dx = \ln 2$, so the identity holds. Suppose it holds for n . By the definition of A_n and part (b),

$$\begin{aligned}\ln 2 - A_{n+1} &= \left(-\frac{1}{2}\right)^n F_n - \frac{3}{4} \left(-\frac{1}{2}\right)^n B_n \\ &= \left(-\frac{1}{2}\right)^n \left(F_n - \frac{3}{4} B_n\right) \\ &= \left(-\frac{1}{2}\right)^{n+1} F_{n+1}.\end{aligned}$$

Thus the result follows by induction.

Marking: Verify the $n=0$ base case and set up a valid induction (or equivalent telescoping proof). | Complete the induction using the recurrence, with correct signs.

2017 mapping: MEX12-1, MEX12-2, MEX12-5, MEX12-7, MEX12-8; MEX-C1, MEX-P1, MEX-P2.

(d) 2 marks

The integrand defining F_n is positive for $0 < x < 1$, so $F_n > 0$. The sign of the error in part (c) therefore gives

$$A_{2n} < \ln 2 < A_{2n+1}.$$

Since $0 \leq x(1-x) \leq 1/4$,

$$0 < F_n \leq \left(\frac{1}{4}\right)^n \int_0^1 \frac{dx}{1+x} = 4^{-n} \ln 2.$$

Consequently

$$|\ln 2 - A_n| = 2^{-n} F_n \leq 8^{-n} \ln 2 \longrightarrow 0,$$

which proves the required limit.

Marking: Use strict positivity of F_n and part (c) to establish both strict inequalities. | Obtain a geometric bound tending to zero and deduce convergence. Merely citing alternating signs is insufficient.

2017 mapping: MEX12-1, MEX12-2, MEX12-5, MEX12-7, MEX12-8; MEX-C1, MEX-P1, MEX-P2.

(e) 5 marks

By part (c), the quantity inside braces equals $F_n/B_n - 2/3$. Put $q = x(1 - x)$ and $y = 1 - 4q = (2x - 1)^2$. Part (b) gives

$$\frac{F_n}{B_n} - \frac{2}{3} = \frac{1}{6B_n} \int_0^1 \frac{q^n y}{2 + q} dx.$$

Using the supplied identity and then part (a),

$$\begin{aligned} \frac{F_n}{B_n} - \frac{2}{3} &= \frac{2}{27B_n} \int_0^1 q^n y dx + \frac{1}{54B_n} \int_0^1 \frac{q^n y^2}{2 + q} dx \\ &= \frac{2}{27(2n + 3)} + \frac{1}{54B_n} \int_0^1 \frac{q^n y^2}{2 + q} dx. \end{aligned}$$

Part (a), applied at n and at $n + 1$, implies

$$\begin{aligned} \int_0^1 q^n y^2 dx &= \int_0^1 q^n y dx - 4 \int_0^1 q^{n+1} y dx \\ &= \frac{B_n}{2n + 3} - \frac{4B_{n+1}}{2n + 5}. \end{aligned}$$

Since part (a) also gives $B_{n+1} = (n + 1)B_n/[2(2n + 3)]$, this becomes

$$\int_0^1 q^n y^2 dx = \frac{3B_n}{(2n + 3)(2n + 5)}.$$

Now $2 \leq 2 + q \leq 9/4$. The remaining integral is positive and at most one-half of $\int_0^1 q^n y^2 dx$. Thus

$$0 < \left(\frac{F_n}{B_n} - \frac{2}{3} \right) - \frac{2}{27(2n + 3)} \leq \frac{1}{36(2n + 3)(2n + 5)}.$$

Multiplying by n , the lower comparison tends to $1/27$, and the difference tends to 0. The required limit is therefore

$$\boxed{\frac{1}{27}}.$$

All estimates are for real nonnegative integrands; no interchange of an integral and a limiting operation is needed.

Marking: Reduce the limit using part (c) and obtain the integral expression for $F_n/B_n - 2/3$ from part (b). | Apply the supplied identity and part (a) to separate $2/[27(2n+3)]$ from the nonnegative remainder. | Use part (a) at successive indices to obtain the second-moment identity with value $3B_n/[(2n+3)(2n+5)]$. | Bound the nonnegative remainder by a constant multiple of $1/[(2n+3)(2n+5)]$. | Use the resulting bounds to obtain $1/27$. Correct equivalent estimates earn full credit.

2017 mapping: MEX12-1, MEX12-2, MEX12-5, MEX12-7, MEX12-8; MEX-C1, MEX-P1, MEX-P2.